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G_2-structures with torsion from half-integrable nilmanifolds

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Abstract

The equations for a G_2-structure with torsion on a product $M^7 = N^6 \times S^1$ are studied in relation to the induced SU(3)-structure on N^6. All solutions are found in the case when the Lee-form of the G_2-structure is non-zero and N^6 is a six-dimensional nilmanifold with half-integrable SU(3)-structure. Special properties of the torsion of these solutions are discussed.

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1. Introduction

Connections with torsion have been objects of geometrical study for many years. Interest in this subject has been increased by considerations from supersymmetric string- and **M**-theories [45], where connections with skew-symmetric torsion coming from G-structures distinguished by spinors play an important role. Recent mathematical discussion of such

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connections may be found, for example, in [22,2,27,28], where Hermitian manifolds, G_2, Spin(7) and quaternionic geometries occur. Particular importance is attached to geometries in dimensions 6 and 7 given by SU(3)- and G_2-structures.

In this paper we investigate the explicit construction of G_2 geometries with torsion (G_2T-structures) from products with six-dimensional SU(3)-manifolds. The initial data on N^6 is an almost Hermitian structure (J, h, ω) together with a distinguished complex volume $\Psi = \psi^+ + \mathrm{i}\psi^-$. A G_2-structure may then be built on the product $M^7 = N^6 \times S^1$ by using the three-form $\varphi = \omega \wedge \mathrm{d}t + \psi^+$.

Following Gray and Hervella [29], one approach to the study of metric G-structures in general is via consideration of the components of the intrinsic torsion in the irreducible G-modules of $T^*M \otimes \mathfrak{g}^\perp$. For G_2, this space splits into four components $\mathcal{X}_i$, $i = 1, \ldots, 4$ [19] and the conditions on φ for the intrinsic torsion $\tau^{(M)}$ to lie in a given combination of these spaces have been determined explicitly in [39]. The pair (M^7, φ) is a G_2T-structure exactly when the intrinsic torsion has no component in $\mathcal{X}_2 \cong \mathfrak{g}_2$, i.e.,

$$\tau^{(M)} \in \mathcal{X}_1 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4. \tag{1.1}$$

Slightly misleadingly, such structures are sometimes referred to as ‘integrable G_2-structures’ in the literature, despite the fact that the Riemannian holonomy need not reduce. Special cases of this geometry are studied in [12,8,24,20].

The corresponding refinement of the Gray–Hervella classification for SU(3)-structures was computed in [10]. Our first task, in Section 2, is to relate the two decompositions on N^6 and M^7 concentrating on the situation for G_2T-structures and refining results of [11].

We then turn to consideration of particular examples. In [21], six-dimensional nilmanifolds were successfully used to give examples of SKT structures: KT geometry is ‘Kähler with torsion’ and consists of a Hermitian manifold (N, J, h, ω) together with its Bismut connection, the unique Hermitian connection with totally skew-symmetric torsion (essentially $J\,\mathrm{d}\omega$), see [25]; the structure is ‘strong’ (SKT) when the torsion is a closed form, i.e., $\mathrm{d}J\,\mathrm{d}\omega = 0$. A nilmanifold is a compact quotient of a nilpotent Lie group. In dimension 6, each nilpotent Lie algebra has a basis with rational structure coefficients and so [38] the corresponding 1-connected group G admits a co-compact lattice Γ. Much is known about the topology [40] and geometry of these manifolds. In particular, they do not admit Kähler metrics unless G is Abelian [5,15,9], a fact related to the theory of minimal models [17,30], and several authors have studied their complex, Hermitian and symplectic geometry [1,14,44,43,16].

We thus study the case of $N^6 = \Gamma\backslash G$ with an SU(3)-structure that pulls back to an invariant SU(3)-structure on G. However, instead of looking at the full system of equations for a G_2T-structure on $M^7 = N^6 \times S^1$, we consider a special case when N^6 is *half-integrable*, see Section 3. Restricting to the case where J is integrable is too severe, we merely obtain structures with $\tau^{(M)} \in \mathcal{X}_1 \oplus \mathcal{X}_4$. The half-integrability condition considered here and in [11] is a weaker restriction on the SU(3)-structure that is of interest in its own right: these are exactly the SU(3)-structures that appear as hypersurfaces in manifolds of holonomy G_2 (Joyce manifolds) and indeed Hitchin [31,32] has shown how the holonomy metric may be obtained by a flow on the space of SU(3)-structures.

In Section 4, we give a full classification of the invariant half-integrable nilmanifolds $N^6 = \Gamma\backslash G$ and their SU(3)-structures such that $M^7 = N^6 \times S^1$ carries a G_2T-structure.

The proof relies on a detailed study of compatibility between the nilpotent algebra structure and the SU(3)-geometry and is facilitated by consideration of a complex version of the equations. A key ingredient to finding concrete solutions is the classification of six-dimensional nilpotent Lie algebras as presented in [43]. In our situation we find six different non-trivial Lie algebras from the classification; in each case the relevant half-integrable SU(3)-structures on G are parameterised by at most two essential variables, see Theorem 4.11. In fact, these six algebras are closely related to each other, and all are obtainable as degenerations of a single example.

We then analyse in Section 5 the special properties of the derivative of T for the G_2-structures obtained. We note examples where the Kähler form ω on N^6 is a eigenform for the Laplacian. The eigenform property of ω is shared by 'strong' G_2T-structures of the type suggested by the physical literature, but for a different eigenvalue. Interestingly, one of our examples also occurs in [34] as an example of an 'instanton'. Future work will concentrate on determining properties of the G_2-holonomy metrics containing these SU(3)-structures on hypersurfaces.

2. G_2-structures with torsion

Denote by N a manifold of (real) dimension 6 with a U(3)-structure and trivial canonical bundle. N is thus equipped with an orthogonal almost complex structure J and a non-degenerate 2-form ω. The induced Riemannian metric h distinguishes the circle consisting of elements of unit norm in the two-dimensional space $[\![\Lambda^{3,0}]\!]$, and an SU(3)-structure is determined by the choice of a real 3-form ψ^+ lying in this S^1-bundle at each point. The associated (3, 0)-form is $\Psi = \psi^+ + \mathrm{i}\psi^-$ with $\psi^- = J\psi^+$, and the knowledge of the tensors J, ω, Ψ determines the geometry of the manifold in full, though only ω, ψ^+ are, strictly speaking, necessary to pin down the reduction to SU(3).

Due to the local nature of the set-up, all descriptions will be valid at least pointwise, so by choosing an orthonormal basis $e^1, \ldots, e^6$ for the cotangent bundle T^*N, we define the following forms

$$\begin{aligned}
&\omega = e^{12} + e^{34} + e^{56} \in \Lambda^{1,1}N,\\
&\psi^+ + i\psi^- = (e^1 + \mathrm{i}e^2) \wedge (e^3 + \mathrm{i}e^4) \wedge (e^5 + \mathrm{i}e^6) \in \Lambda^{3,0}N,
\end{aligned} \tag{2.1}$$

relative to the decomposition into types determined by J. We will freely use the familiar notation $e^{ij\cdots}$ to indicate $e^i \wedge e^j \wedge \cdots$, so for instance $\psi^+ = e^{135} - e^{146} - e^{236} - e^{245}$ and $\psi^- = e^{136} + e^{145} + e^{235} - e^{246}$. In an even more concise way we shall sometimes write, say the Kähler form, as $\omega = 12 + 34 + 56$.

The classical Gray–Hervella decomposition of the Hermitian intrinsic torsion space into four irreducible representations $\mathcal{W}_k$, $k = 1, \ldots, 4$, was extended to tackle SU(3) reductions in [11] (but see also [6]). In the former some relations between G_2 manifolds and underlying SU(3)-structures were taken into consideration, and we shall adhere to the same notation throughout. While retaining curly symbols for the almost Hermitian classes, W_j denotes the corresponding intrinsic torsion component. The new elements in the theory are the presence of an extra fifth class $\mathcal{W}_5$ whose component depends on $(\mathrm{d}\psi^+)^{3,1}$, and the reducibility of the $U(3)$-modules $\mathcal{W}_1, \mathcal{W}_2$ which split into symmetric halves, denoted by $\mathcal{W}_1^{\pm}, \mathcal{W}_2^{\pm}$. This allows one to introduce a correspondence between hypersurfaces of G_2-manifolds and S^1-quotients, a fact reflected in the self-duality versus anti-self-duality picture in four dimensions [11].

We consider a Riemannian product M^7 of N with a circle, endowed with product metric g. By means of the almost Hermitian structure the manifold $M = N \times S^1$ naturally inherits a G_2-structure by declaring

$$\varphi = \omega \wedge \mathrm{d}t + \psi^+ \in \Lambda^3 T^* M. \tag{2.2}$$

In the terminology of [32] this three-form is stable and defines a reduction of the structure bundle to the exceptional group G_2. Though relying on standard references [7,42,35] for the theory of G_2-structures, we recall here the fundamental fact that a form of the kind (2.2) completely specifies the Riemannian metric g, an orientation and Hodge operator $*$. The basis $e^1, \ldots, e^6, e^7 = \mathrm{d}t$ is also orthonormal for the metric g determined by the inclusion $G_2 \subset \mathrm{SO}(7)$, and one has $\varphi = 127 + 347 + 567 + 135 - 146 - 236 - 245$.

It is an easy matter of calculation to see that condition (1.1) holds if and only if

$$\mathrm{d}{*\varphi} = \theta \wedge *\varphi \tag{2.3}$$

for some 1-form θ (see [39]), poignantly called by Friedrich and Ivanov the Lee form of the G_2-structure [23]. We shall be mainly concerned with the case where θ does not vanish. The particular interest of this class is clear by the result shown in the following theorem.

Theorem 2.1 (Friedrich and Ivanov[22]). *On a G_2-manifold (M, φ) the following are equivalent*:

(1) *the torsion of the G_2-structure is a three-form $T \in \Lambda^3 T^* M$;*
(2) *there exists a unique linear connection with skew-symmetric torsion*

$$T = \tfrac{1}{6}\langle \mathrm{d}\varphi, *\varphi\rangle \varphi - *\mathrm{d}\varphi + *(\theta \wedge \varphi), \tag{2.4}$$

where $\langle\, ,\, \rangle$ is the inner product given by the metric.

Reflecting the orthogonal splitting $T^* M = \mathbb{R}^6 \oplus \mathbb{R}$, we decompose

$$\theta = \beta + \lambda\, \mathrm{d}t \tag{2.5}$$

for some $\lambda \in \mathbb{R}$ and 1-form $\beta \in \Lambda^1 N^6$. One immediate consequence of (2.3) is that the Lee form is closed, for the wedging map with $*\varphi$ is a one-to-one homomorphism between $\Lambda^2 = \mathbb{R}^7 \oplus \mathfrak{g}_2$ and $\Lambda^6 = \Lambda^1 = \mathbb{R}^7$. The effects of this fact in physics are known, and will be recalled later. The class of G_2-manifolds with closed Lee form is known as that of locally conformally balanced G_2-structures. The term 'balanced' reflects the six-dimensional set-up with the same name, where balanced, or cosymplectic, refers to a Hermitian structure with $\vartheta = -J\, \mathrm{d}^* \omega = 0$ (which is called Lee form, too). In terms of intrinsic torsion τ, this amounts to $\tau^{(N)} \in \mathcal{W}_3$, echoed in seven dimensions by co-calibrated G_2-structures, for which $\tau^{(M)}$ belongs to $\mathcal{X}_1 \oplus \mathcal{X}_3$.

We begin investigating the geometric properties of (2.3) by expanding the exterior derivative of $*\varphi = \psi^- \wedge \mathrm{d}t + (1/2)\omega \wedge \omega$ into

$$\mathrm{d}{*\varphi} = \mathrm{d}\psi^- \mathrm{d}t + \omega\, \mathrm{d}\omega = \beta\psi^-\, \mathrm{d}t + \tfrac{1}{2}\beta\omega^2 + \tfrac{1}{2}\lambda\omega^2\, \mathrm{d}t = (\lambda\, \mathrm{d}t + \beta){*\varphi},$$

where we start dropping wedge product signs to lighten expressions. Comparing components yields

$$\mathrm{d}\psi^- = \beta\psi^- + \tfrac{1}{2}\lambda\omega^2, \qquad \omega\, \mathrm{d}\omega = \tfrac{1}{2}\beta\omega^2, \tag{2.6}$$

which we shall refer to as the G_2T equations for N. Since these are equivalent to $\mathcal{X}_2 = 0$, we obtain the first restrictions

Lemma 2.2. *Whenever the G_2-structure of $M = N \times S^1$ has a three-form torsion, the intrinsic torsion of N satisfies*

$$W_2^- = 0, \qquad W_1^- = \tfrac{1}{2}\lambda, \qquad W_5 = -2W_4. \tag{2.7}$$

Proof. An immediate consequence of [11]. □

Remark 2.3. The vanishing of the linear combination $2W_4 + W_5$ is a requirement in four-dimensional $\mathcal{N} = 1$ spacetime supersymmetry to give rise to supersymmetric compactifications of heterotic string theory [45].

The SU(3) components W_4, W_5 are explicitly given by:

$$W_4 = \tfrac{1}{2}\omega\lrcorner\, \mathrm{d}\omega = \tfrac{1}{4}\beta \quad \text{so that } W_5 = -\tfrac{1}{2}\beta.$$

Let us decompose $\mathrm{d}\psi^+$ into types and set $\mathrm{d}\psi^+ = \gamma\psi^+ + W_2^+\omega + W_1^+\omega^2$. The definition of $W_5 = (1/2)\psi^+\lrcorner\, \mathrm{d}\psi^+$ tells us that $\gamma = -\beta$, so

$$\mathrm{d}\psi^+ = -\beta\psi^+ + W_2^+\omega + W_1^+\omega^2.$$

We now repeat the procedure for the derivative of the Kähler form, initially prescribing $\mathrm{d}\omega = (1/2)\beta\omega + \Omega + \nu^+\psi^+ + \nu^-\psi^-$, for $\nu^\pm \in \mathbb{R}$. Here $\Omega = (\mathrm{d}\omega)_0^{2,1}$ represents the component in $\mathcal{W}_3$. In local coordinates, both ω^3 and $\psi^+ \wedge \psi^-$ are multiples of the volume form:

$$\omega^3 = 6e^{12\cdots 6}, \qquad \psi^+\psi^- = 4e^{12\cdots 6}$$

in agreement with the compatibility equation $\psi^+\psi^- = (2/3)\omega^3$ (cf. [31]). Using the definition of $W_1^\pm$, we find $W_1^\pm\omega^3 = \mathrm{d}\psi^\pm \wedge \omega = \psi^\pm \wedge \mathrm{d}\omega = 6W_1^\pm e^{12\ldots 6}$, and $\psi^\pm \wedge \omega = 0$ follows by type considerations, whence

$$\psi^+ \wedge \mathrm{d}\omega = \nu^-\psi^+\psi^- \Longrightarrow \nu^- = \tfrac{3}{2}W_1^+, \quad \psi^- \wedge \mathrm{d}\omega = \nu^+\psi^-\psi^+ \Longrightarrow \nu^+ = -\tfrac{3}{4}\lambda.$$

At last then, we are able to express

$$\mathrm{d}\omega = \tfrac{1}{2}\beta\omega + \omega - \tfrac{3}{4}\lambda\psi^+ + \tfrac{3}{2}W_1^+\psi^-.$$

Aiming at the expression for the totally skew-symmetric torsion, we determine each term of (2.4) separately, starting by computing

$$\begin{aligned} \mathrm{d}\varphi &= \mathrm{d}\omega\, \mathrm{d}t + \mathrm{d}\psi^+ \\ &= \tfrac{1}{2}\beta\omega\, \mathrm{d}t + \omega\, \mathrm{d}t + \tfrac{3}{2}W_1^+\psi^-\, \mathrm{d}t - \tfrac{3}{2}W_1^-\psi^+\, \mathrm{d}t - \beta\psi^+ + W_2^+\omega + W_1^+\omega^2, \end{aligned}$$

whence $\langle \mathrm{d}\varphi, *\varphi\rangle = (3/2)W_1^+\|\psi^-\|^2 + (1/2)W_1^-\|\psi^+\|^2 = 12W_1^+$.

Remark 2.4. This last term has a relevant physical meaning: its vanishing is precisely the Killing spinor equation $\nabla\eta = 0$ prescribing the existence of a parallel spinor field η with respect to the torsion connection (consult [23,45]). There is a second equation akin to this, namely $\theta = -2\,\mathrm{d}\phi$, where ϕ represents the dilation function of string theory. We have already shown that θ is closed (so locally exact) in the present setting, see (2.5).

Paying a little attention to the different behaviour of the Hodge star operators, denoted $*_6$ if acting on $\mathbb{R}^6$, we have

$$\begin{aligned} *\mathrm{d}\varphi &= \tfrac{1}{2}*(\beta\omega\, \mathrm{d}t) + *(\omega\, \mathrm{d}t) + \tfrac{3}{2}W_1^-\psi^- + \tfrac{3}{2}W_1^+\psi^+ - *(\beta\psi^+) \\ &\quad + *(W_2^+\omega) + 2W_1^+\omega\, \mathrm{d}t \\ &= -\tfrac{1}{2}J\beta\wedge\omega + *_6\omega + \tfrac{3}{2}W_1^-\psi^- + \tfrac{3}{2}W_1^+\psi^+ + \beta^\sharp \lrcorner\, \psi^-\, \mathrm{d}t \\ &\quad + *(W_2^+\omega) + 2W_1^+\omega\, \mathrm{d}t. \end{aligned}$$

Eventually, using the G_2 Lee form (2.5),

$$*(\theta\wedge\varphi) = \lambda\psi^- - J\beta\wedge\omega - \beta^\sharp\lrcorner\,\psi^-\, \mathrm{d}t.$$

Collecting the relevant terms we rewrite (2.4) as follows:

$$T = -\tfrac{1}{2}J\beta\wedge\omega - 2\beta^\sharp\lrcorner\,\psi^-\, \mathrm{d}t - *_6\Omega - *(W_2^+\omega) + \tfrac{1}{2}W_1^-\psi^- + \tfrac{1}{2}W_1^+\psi^+,$$

and expressing $*_6\Omega$ in terms of $\mathrm{d}\omega$ we arrive at the more concise

$$T = -2\beta^\sharp\lrcorner\,\psi^-\, \mathrm{d}t - *(W_2^+\omega) - *_6\,\mathrm{d}\omega + 2W_1^+\psi^+ - \tfrac{1}{2}\lambda\psi^-. \tag{2.8}$$

To have more readable formulæ we shall denote the 1-form β and its dual vector $\beta^\sharp$ by the same symbol.

Furthermore, we shall refer to

$$*T = 2\beta\psi^+ - W_2^+\omega + \mathrm{d}\omega\, \mathrm{d}t + 2W_1^+\psi^-\, \mathrm{d}t + \tfrac{1}{2}\lambda\psi^+\, \mathrm{d}t \tag{2.9}$$

in the sequel, and to the differentials

$$\begin{aligned} \mathrm{d}T &= -\mathrm{d}*_6\,\mathrm{d}\omega - 2W_1^+\beta\psi^+ + 2W_1^+W_2^+\omega - \tfrac{1}{2}\lambda\beta\psi^- + \big(2(W_1^+)^2 - \tfrac{1}{4}\lambda^2\big)\omega^2 \\ &\quad - 2\mathrm{d}(\beta\lrcorner\,\psi^-)\,\mathrm{d}t - \mathrm{d}*(W_2^+\omega) \\ \mathrm{d}*T &= -2\mathrm{d}(W_2^+\omega) + \tfrac{1}{2}\lambda W_2^+\omega\, \mathrm{d}t + 2W_1^+\beta\psi^-\, \mathrm{d}t - \tfrac{1}{2}\lambda\beta\psi^+\, \mathrm{d}t + \tfrac{3}{2}\lambda W_1^+\omega^2\, \mathrm{d}t \\ &\quad - 2\beta W_2^+\omega - 2W_1^+\beta\omega^2. \end{aligned} \tag{2.10}$$

Example 2.5. Let N be the Iwasawa manifold, that is, the compact quotient of the complexified Heisenberg group $G = \mathcal{H}_3{}^{\mathbb{C}}$ by the sublattice of matrices with Gaussian integer entries. This nilmanifold arises from the Lie algebra $\mathfrak{g}$ of G given by

$$\mathrm{d}e^i = (0, 0, 0, 0, 13 + 42, 14 + 23).$$

Endow N with the non-integrable almost complex structure J_3, in the notation of [1], and select the following invariant SU(3)-structure

$$\omega_3 = -12 - 34 + 56, \qquad \psi^+ = 135 + 146 + 236 - 245.$$

Whilst the G_2T equations reduce to $\mathrm{d}\psi^- = 0$, $\mathrm{d}\omega^2 = 0$, all geometric information is determined by

$$\mathrm{d}\psi^+ = 4e^{1234} \in \Lambda^{2,2}, \qquad \mathrm{d}\omega_3 = \psi^-.$$

It is known that the G_2-structure $\varphi = \omega_3 \wedge e^7 + \psi^+$ is co-calibrated, and in fact $\mathrm{d}{*\varphi} = \theta \wedge *\varphi = 0$ implies $\theta = 0$, thus $\beta = 0$, $\lambda = 0$.

For later purposes, notice that the three-form torsion is given by $T = (2/3)\varphi - 4e^{567}$, hence in particular is not closed.

3. Integrable and half-integrable structures

When the Nijenhuis tensor of an almost Hermitian manifold N is zero, the canonical torsion connection is the Bismut connection and the G_2T equations reduce to

$$\mathrm{d}\psi^- = \beta\psi^-, \qquad \omega\,\mathrm{d}\omega = \tfrac{1}{2}\beta\omega^2.$$

The intrinsic torsion components (possibly) surviving are

$$W_3 = \Omega, \qquad W_4 = \tfrac{1}{4}\beta, \qquad W_5 = -\tfrac{1}{2}\beta,$$

the remaining data being encoded by $\mathrm{d}\omega = \Omega + (1/2)\beta\omega$.

The nasty expression (2.8) now becomes the much more tractable

$$T = -*_6\,\mathrm{d}\omega - 2\beta\lrcorner\,\psi^-\,\mathrm{d}t,$$

and a glance at (2.9) shows the following corollary.

Corollary 3.1. *When (N, J) is a complex manifold, the tensor T is co-closed.*

This fact was implicitly noticed in [22].

But our investigation focuses on six-dimensional nilmanifolds: they have this striking property that makes all the above relations completely trivial. The holomorphic section Ψ of the canonical bundle can be expressed locally as product of (1,0)-forms α^i – given an SU(3)-structure such that J is complex – with $\mathrm{d}\alpha^i \in \mathcal{I}(\alpha^1, \dots, \alpha^{i-1})$ [43]. Thus $\mathrm{d}\Psi = 0$. Conversely, if (N, J, ψ^+) is a nilmanifold and ψ^+, ψ^- are closed then $(\mathrm{d}\alpha^i)^{0,2} = 0$, hence $\mathrm{d}\Lambda^{1,0} \subseteq \Lambda^{2,0} \oplus \Lambda^{1,1}$. Since $\nabla\psi^\pm = \mp\beta \otimes \psi^\pm$ by hypothesis, the 1-form β vanishes identically. What is more, the Lee form $\vartheta = -J\,\mathrm{d}^*\omega = -\beta$ ($\mathrm{d}^* = -{*_6}\,\mathrm{d}{*_6}$ being the formal adjoint of the exterior differential) is zero as well, so only the $\mathcal{W}_3$-component of the intrinsic torsion remains:

Proposition 3.2. *An* SU(3)-*nilmanifold* (N, ψ^+, ω) *satisfying the* G_2T *equations* (2.6) *and having an integrable almost complex structure is necessarily balanced.*

A somehow simpler situation will crop up in Section 5, where $(\mathrm{d}\omega)_0^{2,1}$ will vanish too, and the manifold will turn out to be Kähler. So on products of the type $N \times S^1$ the integrability hypothesis proves very restrictive and one problem is to understand the case where the connection forms on N and S^1 behave in a more entangled way.

Having seen that $N_J = 0$ forces the G_2 Lee form to vanish and that the G_2T-structure is no longer of the general type, we consider another distinguished setting which is not so restrictive, namely that of half-integrability.

Definition 3.3 (Chiossi and Salamon [11]). An almost Hermitian 6-manifold is *half-integrable* if it possesses a reduction to SU(3) for which both ψ^+ and ω^2 are closed.

Although this kind of structure has become popular in a certain part of the physical literature with the name ‘half-flat’, it might be preferable to refer to such geometry with the term half-integrable, as in [4]. When G_2-manifolds are built from SU(3)-structures, many features of the G_2-structure are obtained by properties in six dimensions, and the way an admissible 3-form is built motivated the above definition. This turns out to be a useful notion especially in connection to Hitchin’s evolution equations [32] which give metrics with holonomy G_2 in dimension 7 starting from any such half-integrable structure in dimension 6 [10].

In terms of SU(3) classes half-integrability amounts to

$$W_1^+ = 0, \qquad W_2^+ = 0, \qquad W_4 = 0 = W_5.$$

The hypothesis $\lambda \neq 0$ in (Eq. 2.7) is necessary to avoid integrability issues, which we have already dealt with, so the relevant non-vanishing derivatives are

$$\mathrm{d}\psi^- = \tfrac{1}{2}\lambda\omega^2, \qquad \mathrm{d}\omega = \Omega - \tfrac{3}{4}\lambda\psi^+.$$

For the sake of completeness we also write down

$$T = -{*_6}\,\mathrm{d}\omega - \tfrac{1}{2}\lambda\psi^-, \qquad *T = (\mathrm{d}\omega + \tfrac{1}{2}\lambda\psi^+)\,\mathrm{d}t, \tag{3.1}$$

in accordance with the fact that $T = 0$ implies the reduction of the holonomy.

While leaving the discussion of the properties of $\mathrm{d}T$ to Section 5, we are able now to weaken Corollary 3.1 a little.

Corollary 3.4. *If* (N, ω, ψ^+) *is half-integrable then* T *is co-closed.*

Notice that the Lee form ϑ of the SU(3)-structure vanishes and the G_2 Lee form θ becomes locally exact, as required by the Killing spinor equation mentioned in Remark 2.4.

Remark 3.5. A different way to simplify (2.6) is to assume the closure of ψ^+ only. But this annihilates β (since $\beta \wedge \psi^+ = 0 \Longrightarrow \beta = 0$) and makes the $\mathcal{W}_1 \oplus \mathcal{W}_2$-component vanish identically, so in the present context $\mathrm{d}\omega^2 = 0$ follows from $\mathrm{d}\psi^+ = 0$. In other words we land on half-integrability once again.

4. Half-integrable nilpotent Lie algebras

The G_2T equations combined with half-integrability form a powerful set of constraints

$$\mathrm{d}\psi^- = \frac{1}{2}\lambda\omega^2, \qquad \mathrm{d}\psi^+ = 0, \tag{4.1}$$

and the aim is to try to detect all half-integrable nilpotent Lie algebras generating G_2T-structures in seven dimensions in the described way. We remind the reader that $\lambda \neq 0$ is the overall assumption from now onwards.

Consider a six-dimensional nilpotent Lie algebra $\mathfrak{g}$. Any such has a nilpotent basis (E^i), i.e. a basis of 1-forms such that

$$\mathrm{d}E^i \in \Lambda^2 V_{i-1}, \quad \text{where } V_j = \mathrm{span}_{\mathbb{R}}\{E^1, \ldots, E^{j-1}\},$$

and the spaces V_j filter the Lie algebra: $0 \subset V_1 \subset \cdots \subset V_5 \subset V_6 = \mathfrak{g}^*$. We shall indicate the basis $E^1, \ldots, E^6$ informally by $1, \ldots, 6$ when no confusion arises.

Remark 4.1. By adjusting the above spaces, it is always possible to take V_i to be the kernel of d, for some i, hence assume $V_2 \subseteq \operatorname{Ker} \mathrm{d}$. This small observation underpins the relation between the algebraic nilpotent filtration and the geometry under study.

In order to find all possible SU(3)-structures satisfying our equations we need to determine a different orthonormal basis $e^1, \ldots, e^6$ of $\mathfrak{g}^*$ for which the Kähler form ω and the holomorphic volume Ψ are those of (2.1).

Let $U_j = V_j \cap JV_j$ be the subspaces corresponding to the V_j's invariant for the almost complex structure, so

$$\begin{array}{ccccccc} 0 \neq & U_4 & \subseteq & U_5 & \subset & U_6 \\ & \cap\!\!| & & \cap & & \| \\ & V_4 & \subset & V_5 & \subset & V_6. \end{array}$$

Since $\mathrm{d}U_j \subseteq \Lambda^2 V_{j-1}$, a dimension count tells that U_5 has real dimension 4, and more importantly

$$2 = \dim_{\mathbb{C}} U_4 \iff JV_4 = V_4,$$

that is U_4 is maximal if and only if V_4 is J-invariant. This will be the first major watershed of the discussion. Since $\dim(U_5^\perp \cap V_5) = 1$ we also have $\dim_{\mathbb{C}} U_3, \dim_{\mathbb{C}} U_2 \in \{0, 1\}$ and $\dim_{\mathbb{C}} U_1 = 0$, whence U_2 has (complex) dimension 1 exactly if $JV_2 = V_2$.

Let us fix the SU(3)-structure (2.1) and concentrate only on the hypothesis $V_4 = U_4$, and prove the following lemma.

Lemma 4.2. *For any nilpotent Lie algebra, under the assumption* $V_4 = JV_4$ *there is no solution to* $\mathrm{d}\psi^- = (\lambda/2)\omega^2$, $\mathrm{d}\psi^+ = 0$ (*when* $\lambda \neq 0$).

Proof. One can assume

$$e^1, e^2, e^3, e^4 \in U_4, \qquad e^5 \in V_5 \cap U_4^\perp, \qquad e^6 \in JV_5 \cap U_4^\perp.$$

Since $\mathfrak{g}^* = V_5 \oplus \langle e^6 \rangle$, from $\mathrm{d}\psi^- = (\lambda/2)\omega^2$ we induce $\mathrm{d}(e^{13} + e^{42}) = -\lambda(e^{34} + e^{12})e^5$; but while the left-hand side of the latter lives in $\Lambda^3 V_4$, $e^5 \in V_5$ so the other side belongs to $\Lambda^2 V_4 \wedge V_5$, and in particular $\lambda = 0$. □

Example 4.3.

(a) The Iwasawa manifold once more provides a simple instance. As $\dim_{\mathbb{R}} \operatorname{Ker} d|_{\Lambda^3} = 15$, the space of complex structures has dimension 12 (see [36]), so the closure of ψ^+ entails that $\mathrm{d}\psi^-$ is proportional to 1234, and the G_2T equations do not hold. It is no coincidence that it is precisely the 'λ equation' that fails.

(b) More generally, for all Lie algebras with $\mathrm{d}V_4 = 0$ something similar happens. $V_4^\perp$ is J-invariant too, and there is a basis such that $V_4 = \operatorname{span}\{e^1, \dots, e^4\}$, $V_4^\perp = \operatorname{span}\{e^5, e^6\}$. Let

$$\alpha^1 = e^1 + \mathrm{i}e^2, \qquad \alpha^2 = e^3 + \mathrm{i}e^4, \qquad \alpha^3 = e^5 + \mathrm{i}e^6$$

be a basis of complex (1, 0)-forms determined by the almost complex structure J, so we may write

$$\Psi = \alpha^1 \wedge \alpha^2 \wedge \alpha^3 = \alpha^{123}.$$

Since $\mathrm{d}\alpha^3 \in \Lambda^2 \langle \alpha^1, \alpha^2, \overline{\alpha^1}, \overline{\alpha^2} \rangle$

$$\mathrm{d}\Psi = \alpha^1 \wedge \alpha^2 \wedge \mathrm{d}\alpha^3 \in \Lambda^4 V_4 \otimes \mathbb{C} = \operatorname{span}_{\mathbb{C}}\{e^{1234}\}.$$

But $(1/2)\omega^2 = e^{1234} + e^{1256} + e^{3456}$ does definitely not belong to the latter, so Eq. (4.1) cannot be solved unless $\lambda = 0$. A somehow more elegant proof of this fact will follow from Lemma 4.4.

The possibility that $\dim_{\mathbb{C}} U_4 = 1 < \dim_{\mathbb{C}} U_5$ involves more thinking. Let us change perspective for a moment and note that for an invariant SU(3) structure equations (4.1) necessarily have λ constant and so can be rewritten as the single equation

$$\mathrm{d}\Psi = \tilde{\lambda}\omega^2 \tag{4.2}$$

with $\tilde{\lambda} = \mathrm{i}\lambda/2$. One may now regard this as an equation for a $U(3)$-structure and allow $\tilde{\lambda}$ to be a complex number.

The symbol $\alpha^{\bar{\imath}}$ will denote $\overline{\alpha^{\imath}}$ for $i = 1, 2, 3$. Since $\Psi = \alpha^{123}$ and

$$-2\omega^2 = \alpha^{1\bar{1}2\bar{2}} + \alpha^{1\bar{1}3\bar{3}} + \alpha^{2\bar{2}3\bar{3}},$$

Eq. (4.2) becomes

$$(\mathrm{d}\alpha^1)\alpha^{23} + (\mathrm{d}\alpha^2)\alpha^{31} + (\mathrm{d}\alpha^3)\alpha^{12} = -\tfrac{1}{2}\tilde{\lambda}(\alpha^{1\bar{1}2\bar{2}} + \alpha^{1\bar{1}3\bar{3}} + \alpha^{2\bar{2}3\bar{3}}),$$

which implies immediately

$$(\mathrm{d}\alpha^1)^{0,2} = \tfrac{1}{2}\tilde{\lambda}\alpha^{\overline{23}}, \qquad (\mathrm{d}\alpha^2)^{0,2} = \tfrac{1}{2}\tilde{\lambda}\alpha^{\overline{31}}, \qquad (\mathrm{d}\alpha^3)^{0,2} = \tfrac{1}{2}\tilde{\lambda}\alpha^{\overline{12}}. \tag{4.3}$$

Thus a choice of e.g. α^1 determines the span of $\alpha^{\bar{2}}$ and $\alpha^{\bar{3}}$ in $\Lambda^{0,1}$. Conjugating, one gets the span of α^2 and α^3 in $\Lambda^{1,0}$, and hence the full orthogonal complement of span$\{\alpha^1, \alpha^{\bar{1}}\}$.

As U_4, $V_3 \subset V_4$ and these have real dimensions 2, 3 and 4, respectively, we have

$$\dim U_4 \cap V_3 \geqslant 1.$$

Similarly,

$$\dim U_5 \cap U_4^{\perp} \cap V_4 \geqslant 1, \qquad \dim U_5^{\perp} \cap V_5 \geqslant 1.$$

We may now take

$$\begin{aligned} &\alpha^1 = e^1 + \mathrm{i}e^2 \in U_4 && \text{with } e^1 \in V_3, \\ &\alpha^2 = e^3 + \mathrm{i}e^4 \in U_5 \cap U_4^{\perp} && \text{with } e^3 \in V_4, \\ &\alpha^3 = e^5 + \mathrm{i}e^6 \in U_5^{\perp} && \text{with } e^5 \in V_5. \end{aligned}$$

The first of Eq. (4.3) implies that $\mathrm{d}\alpha^1 \neq 0$, so one of $\mathrm{d}e^1$, $\mathrm{d}e^2$ is non-zero and therefore

Lemma 4.4. *The space of closed* 1*-forms has real dimension at most three, i.e.* $b_1 \leqslant 3$.

Notice by the way that the first Betti number of any nilmanifold is always strictly greater than one [18], which reduces our investigations to the nilpotent Lie algebras with b_1 equal to either 2 or 3. It becomes clear how crucial Remark 4.1 is in the description, for it now states that the kernel of the exterior derivative is V_2 or possibly V_3.

Although the argument of Lemma 4.2 can be adapted to fit Lie algebras with many closed forms, a straightforward consequence of the previous result is that

Corollary 4.5. *For any six-dimensional nilpotent Lie algebra with* $b_1 \geq 4$ *the equations* $\mathrm{d}\psi^- = (\lambda/2)\omega^2$, $\mathrm{d}\psi^+ = 0$ *have no solution, for* λ *non-zero.*

The complex filtration $\{U_\alpha\}$ is rigid in the sense the size of the spaces U_α with odd index is fixed, since $\dim_{\mathbb{C}} U_1 = 0$, $\dim_{\mathbb{C}} U_5 = 2$, and we claim that

Lemma 4.6. $U_3 = \{0\}$ *when* U_4 *is not maximal.*

Proof. If U_3 were non-trivial then $e^1, e^2 \in V_3$ and we might take $e^1 \in V_2$. Then $\mathrm{d}\alpha^1 \in \Lambda^2 V_2$ and $e^1\,\mathrm{d}\alpha^1 = 0$. But

$$(e^1\,\mathrm{d}\alpha^1)^{0,3} = (e^1)^{0,1}(\mathrm{d}\alpha^1)^{0,2} = \tfrac{1}{4}\tilde{\lambda}\alpha^{\overline{123}} \neq 0,$$

yielding a contradiction. □

With $\dim_{\mathbb{C}} U_3 = 0$, we now have

$$e^2 \in V_4 \setminus V_3, \qquad e^4 \in V_5 \setminus V_4, \qquad e^6 \in V_6 \setminus V_5.$$

As $\{e^1, e^2, e^3, e^4, e^5\}$ is a real orthonormal basis for V_5, we conclude

$$e^6 \in V_5^{\perp}.$$

In real components Eq. (4.2) is

$$\mathrm{d}((e^1 + \mathrm{i}e^2)(e^3 + \mathrm{i}e^4)(e^5 + \mathrm{i}e^6)) = 2\tilde{\lambda}(e^{1234} + e^{1256} + e^{3456}),$$

whose e^6-component gives

$$\mathrm{d}((e^1 + \mathrm{i}e^2)(e^3 + \mathrm{i}e^4)) = 2\tilde{\lambda}(e^{12} + e^{34})e^5. \tag{4.4}$$

As $e^4 \in V_5 \setminus V_4$ and $e^5 \in V_5$ we may write

$$e^5 = qe^4 + \xi,$$

for some $q \in \mathbb{R}$ and $\xi \in V_4$. Note that ξ completes $\{e^1, e^2, e^3\}$ to an orthogonal basis for V_4. Eq. (4.4) is now

$$\mathrm{d}((e^1 + ie^2)(e^3 + ie^4)) = 2\tilde{\lambda}((qe^{12} - e^3\xi)e^4 + e^{12}\xi)$$

and extracting the e^4-component we have

$$-\mathrm{d}(e^1 + \mathrm{i}e^2) = 2\tilde{\lambda}(qe^{12} - e^3\xi).$$

However $\mathrm{d}(e^1 + \mathrm{i}e^2) \in \Lambda^2 V_3 \otimes \mathbb{C}$ and as $\dim V_3 = 3$ this two-form is necessarily decomposable. This is true of the right-hand side only if $q = 0$. Therefore we conclude firstly that

$$e^5 \in V_4, \qquad e^4 \in V_5 \cap V_4^{\perp}.$$

Our equation now reads $\mathrm{d}(e^1 + \mathrm{i}e^2) = 2\tilde{\lambda}e^{35}$. Thus some real linear combination of e^1 and e^2 is closed.

Lemma 4.7.

$$de^1 = 0, \qquad de^2 = \lambda e^{35}.$$

Proof. This follows directly from the fact that one of the V_i's is the kernel of d and $\lambda = -i2\tilde{\lambda}$ is real. □

Because $\{e^1, e^2, e^3, e^5\}$ is a basis for V_4, and $e^2 \in V_4 \setminus V_3$, we may write

$$e^3 = c_1 e^2 + \eta_1, \qquad e^5 = c_2 e^2 + \eta_2,$$

with $c_i \in \mathbb{R}$ and η_j linearly independent vectors of V_3. By Lemma (4.7),

$$\Lambda^2 V_3 \ni e^{35} = (c_2\eta_1 - c_1\eta_2)e^2 + \eta_1\eta_2.$$

Thus $c_2\eta_1 - c_1\eta_2 = 0$, which gives $c_1 = 0 = c_2$, and hence

$$e^3, e^5 \in V_3, \qquad e^2 \in V_4 \cap V_3^\perp.$$

In particular, V_3 has orthonormal basis $\{e^1, e^3, e^5\}$ and $V_3^\perp$ has orthonormal basis $\{e^2, e^4, e^6\}$, so $V_3^\perp = JV_3$:

Corollary 4.8.

$$\mathfrak{g}^* = V_3 \oplus JV_3 = \langle e^1, e^3, e^5\rangle \oplus \langle e^2, e^4, e^6\rangle.$$

With this, we are able to pin down the bases for almost every space in the filtration, with the exception of the first two. Indeed

$$V_3 = \langle e^1, e^3, e^5\rangle, \qquad V_4 = V_3 \oplus \langle e^2\rangle, \qquad V_5 = V_4 \oplus \langle e^4\rangle.$$

In the light of Remark 4.1, we could have chosen the filtration so that to fix V_1, V_2 as well. As we shall see at the end of the section, it would have been possible, in theory, to decree V_1 to be the span of e^1, and V_2 to be either $\langle e^1, e^3\rangle$ or $\langle e^1, e^5\rangle$. The computation will indeed show that e^3 and e^5 play interchangeable roles. Apparently though not much is gained from this fact, a reason why we prefer the more general point of view.

We now inspect the space V_3 with regard to the fact that $dV_3 \subseteq \Lambda^2 V_2$. Since $de^3 \in \Lambda^2 V_2$ either e^3 is closed or $b_1 = 2$ (recall that $e^1 \in \operatorname{Ker} d$ anyway), so $e^1 \in V_2$. Whichever the case we may write

$$e^1 de^3 = 0. \tag{4.5}$$

A similar argument holds for $de^5 \in \Lambda^2 V_2$, whence

$$e^1 de^5 = 0. \tag{4.6}$$

These two relations will turn out useful later on.

Let us look at the real and imaginary parts of (4.4), now giving the reassuring

$$d(e^{14}+e^{23})=0, \qquad d(e^{13}+e^{42})=\lambda(e^{12}+e^{34})e^5,$$

or, by means of Lemma 4.7, simply

$$e^1 de^4 + e^2 de^3 = 0, \tag{4.7}$$

$$e^2 de^4 = \lambda e^{125}. \tag{4.8}$$

Besides, from (4.8) we may write $de^4 = -\lambda e^{15} + e^2 \wedge V_3$. Explicitly setting $de^4 = -\lambda e^{15} + e^2 \wedge (ve^1 + ue^3 + te^5)$ gives

$$e^1(ue^3 + te^5) = de^3. \tag{4.9}$$

Since the codimension of V_2 in V_3 is one, some linear combination of e^3 and e^5 is in V_2, hence closed. In fact, fix a non-zero element ν in $\Lambda^2 V_2 = \mathbb{C}$, so that

$$de^3 = x\nu, \qquad de^5 = y\nu \tag{4.10}$$

for some $y, x \in \mathbb{R}$. Then the closure of e^{35} makes $(xe^5 - ye^3)\nu$ vanish, whence

$$xe^5 - ye^3 \in V_2.$$

The equation $\omega\, d\omega = 0$ is a direct consequence of the first in (4.1), so we should not expect to gain new information by manipulating it. Nonetheless, it quickly furnishes substantial results

$$d(e^{12}+e^{34})e^5 + (e^{12}+e^{34})\, de^5 = 0,$$

in which we separate e^4-terms from the remaining ones above and obtain

$$e^{35} de^4 = 0. \tag{4.11}$$

Then (4.9) rephrases as

$$e^1(ue^3 + te^5) = x\nu \tag{4.12}$$

whilst (4.11) implies $v = 0$, so

$$de^4 = -\lambda\, e^{15} + ue^{23} + te^{25}.$$

More is recovered by considering the V_5-component of (4.2), whose real and imaginary parts now become

$$-e^{24}\, de^5 = (e^{14}+e^{23})\, de^6, \qquad -e^{15}\, de^4 + (e^{13}+e^{42})\, de^6 = \lambda e^{1234}. \tag{4.13}$$

Defining

$$de^6 = \alpha \wedge e^4 + \zeta, \qquad \text{with} \quad \zeta \in \Lambda^2 V_4, \quad \alpha \in V_4,$$

the $(e^4)^\perp$- and e^4-terms in (4.13) yield

$$e^{13}\zeta = ue^{1235}, \qquad e^{13}\alpha - e^2\zeta = \lambda e^{123},$$
$$e^{23}\zeta = 0, \qquad e^1\zeta + e^{23}\alpha = -ye^2\nu. \tag{4.14}$$

At this very moment it seems difficult to get much out of these relations. Still we are able to say that the forms ζ and α are of the kind

$$\zeta = z_2e^{12} + z_3e^{13} + z_1e^{23} - ue^{25}, \qquad \alpha = (z_3 - \lambda)e^2 + a_1e^1 + a_3e^3$$

with z_i, a_j real constants. Moreover, the last equation of (4.14) tells that

$$y\nu = -(z_1 + a_1)e^{13} + ue^{15}. \tag{4.15}$$

The constraint $\mathrm{d}(\mathrm{d}e^6) = 0$ forces $z_2 = a_3$, whence

$$\mathrm{d}e^6 = a_1e^{14} + z_1e^{23} - a_3(e^{12} - e^{34}) + z_3(e^{13} - e^{42}) + \lambda e^{42} - ue^{25} \tag{4.16}$$

plus other complicated relations whose full exploitation we will delay.

Given Eqs. (4.5) and (4.6), we are obliged to separate the discussion. It is preferable to handle mutually exclusive cases, so we shall assume that $b_1 = 2$ in the first two cases, allowing three independent closed 1-forms in the third only.

- Case 1: is that in which we suppose that e^3 is not closed.
- Case 2: instead has the derivative of e^3 zero but e^5 non-closed.
- Case 3: deals with both e^3, e^5 being closed (so $b_1 = 3$).

Note that Eqs. (4.12) and (4.15) do not permit e^3, e^5 to be both non-closed. Since $\nu = e^1(xe^5 - ye^3)$ in fact, if x, y were non-zero we would have $u = -xy = xy$, i.e. $e^5 \in V_2 \subseteq \operatorname{Ker} d$.

Case 1. Let k be the non-zero real number such that $\Lambda^2 V_2 \ni \mathrm{d}e^3 = ke^{15}$. The 2-form ν of (4.10) is merely proportional to e^{15}, so from (4.12) we infer that $u = 0$, $t = k$. In addition, by (4.15) we also have $a_1 = -z_1$.

The new structure relations are

$$\mathrm{d}e^3 = ke^{15}, \qquad \mathrm{d}e^4 = -\lambda e^{15} + ke^{25}, \qquad \mathrm{d}e^5 = 0,$$

so the string $(\mathrm{d}e^i)$ only lacks the explicit determination of the last entry. Instead of chasing every little piece of data around, we use (4.14). The demand $\mathrm{d}(\mathrm{d}e^6) = 0$ reduces the number of coefficients in (4.16), in fact $z_3 = \lambda$, $a_3 = 0$ and $a_1 = z_1 (= -a_1)$, thus $\mathrm{d}e^6 = \lambda e^{13}$ and finally the Lie algebra structure is:

$$(0, \lambda e^{35}, ke^{15}, -\lambda e^{15} + ke^{25}, 0, \lambda e^{13}) \tag{4.17}$$

with $\lambda \neq 0 \neq k$.

The next task is to spot this Lie algebra within the list of [43], entries on which will be numbered in bold starting from the top, so for example **28** indicates the Lie algebra of the Iwasawa manifold, **34** that of a torus.

Let us compute the Betti numbers of (4.17): while b_1 is clearly 2, we have that $\dim \operatorname{Ker} \mathrm{d}|_{\Lambda^2} = 8$, so $b_2 = 4$. The only Lie algebra with such cohomological data and decomposable exact 2-forms is number **6**, in other words

$$(0, 0, 12, 13, 23, 14).$$

This could have been checked by modifying the basis in (4.17). In fact, redefining e^4 as $\lambda e^3 + ke^4$ gives $(0, \lambda e^{35}, ke^{15}, k^2e^{25}, 0, \lambda e^{13})$. The successive swaps $e^2 \leftrightarrow e^5, e^6 \leftrightarrow e^4, e^1 \leftrightarrow e^2, e^4 \leftrightarrow e^5$ followed by a flip in the signs of e^1, e^6 lead to **6**, with suitable rescaling.

Case 2. This case will take slightly longer, due to minor complications. Given that $V_2 = \operatorname{span}\{e^1, e^3\}$, Eq. (4.12) provides a neat expression for the derivative of e^4, namely

$$\mathrm{d}e^4 = -\lambda e^{15}.$$

In addition (4.15) determines

$$\mathrm{d}e^5 = -(z_1 + a_1)e^{13}$$

with the assumption that $z_1 + a_1 \neq 0$. When asking $\mathrm{d}e^6$ to be closed, we find that $z_3 = \lambda$, so we may write the Lie algebra as

$$\begin{aligned}&(0, \lambda e^{35}, 0, -\lambda e^{15}, ze^{13},\\ &\quad \tfrac{1}{2}(a_1 + z)(e^{14} + e^{23}) + \tfrac{1}{2}(a_1 - z)(e^{14} - e^{23}) - a_3(e^{12} - e^{34}) + \lambda e^{13}),\end{aligned}$$

where z stands for z_1.

The fact that now $\mathrm{d}e^4$ lives in $\Lambda^2 V_3$ renders the filtration $\{V_i\}$ more flexible to further adjustments. The SU(3)-structure fixes the span of α^3, hence e^5, e^6 are confined by this diagram

$$\begin{array}{ccccccc} V_2 & \subset & V_3 & \subset & V_5 & \subset & V_6 \\ e^1, e^3 & & e^5 & & e^2, e^4 & & e^6. \end{array}$$

Furthermore, if $\mathbb{D}$ denotes the real space $\langle e^1, \ldots, e^4\rangle$, then

$$\Lambda^2\mathbb{D} = \Lambda^2_+ \oplus \Lambda^2_-,$$

where Λ^2_- is generated by the anti-self-dual forms $e^{14} - e^{23}, e^{12} - e^{34}, e^{13} - e^{42}$, and similarly for the self-dual part. Since the decomposition

$$\mathbb{D} = \langle e^1, e^3\rangle \oplus \langle e^2, e^4\rangle$$

has to be preserved, it is possible to act by $SO(2) \subset SU(2)$ on the 2-plane $\langle e^{14} - e^{23}, e^{12} - e^{34}\rangle \subset \Lambda^2_-$ to eliminate the a_3 term above. More explicitly, the required transformation on

$\mathbb{D} = \langle\alpha^1, \alpha^2\rangle_{\mathbb{R}}$ is given by

$$\alpha^1 \mapsto \alpha^{\tilde{1}} = c\alpha^1 + s\alpha^2, \qquad \alpha^2 \mapsto \alpha^{\tilde{2}} = -s\alpha^1 + c\alpha^2,$$

for some rotating constants $c = \cos\sigma$, $s = \sin\sigma$. The common coefficient of $e^{\tilde{1}\tilde{2}}$, $e^{\tilde{3}\tilde{4}}$

$$sc(z - a_1) - a_3(s^2 - c^2)$$

is killed off by a suitable choice of angle, given by $\tan 2\sigma = (a_1 - z)/2a_3$. That allows to simplify the general expression of $\mathrm{d}e^6$, and finally the Lie algebra looks like

$$(0, \lambda e^{35}, 0, -\lambda e^{15}, ze^{13}, a_1 e^{14} - ze^{23} + \lambda e^{13}). \tag{4.18}$$

Now, we need only determine the second Betti number: whilst there are six obvious closed 2-forms, $\mathrm{d}e^{16} = -ze^{123}$ and $\mathrm{d}e^{36} = a_1 e^{134}$ are proportional to $\mathrm{d}e^{25}$, $\mathrm{d}e^{45} \in \mathrm{Im}\,\mathrm{d}|_{\Lambda^2}$, irrespective of the coefficients. Therefore $b_2 = 4$ and (4.18) possibly identifies with **6**, **7**, **8** of Salamon's list.

For instance, $a_1 = 0$ gives **6**, in which every exact 2-form is decomposable. It is easy to find an explicit isomorphism in this case. First, change e^6 to $ze^6 + \lambda e^5$ to get rid of the redundant last term in $\mathrm{d}e^6$, then swap e^2, e^3 in order to have $(0, 0, \lambda e^{25}, -\lambda e^{15}, -ze^{12}, -ze^{23})$. Now, $\mathrm{d}e^4 \wedge \mathrm{d}e^6 = z\lambda e^{1235}$ is the only non-zero wedge product of exact 2-forms (while in **6** $\mathrm{d}5 \wedge \mathrm{d}6 = 1234$), suggesting to swap e^4 and e^5. Then it is a question of exchanging e^3, e^4 and e^1, e^2, reversing the orientation of e^5, e^6 and setting $z = 1$.

As far as $(0, 0, 12, 13, 23, 14 \pm 25)$ are concerned, their characteristic series agree with those of (4.18) regardless of coefficients: the central descending series' dimensions being 6, 4, 3, 1, 0, the derived series' 6, 4, 0 and the upper central series' ones 1, 3, 4, 6. But this is not enough to distinguish the sign, and a more subtle argument has to be given. By analysing **7**, **8** we can say a few things about the generic SU(3) basis, namely that $\mathrm{Ker}\,\mathrm{d} = \langle 1, 2\rangle$, and since $\mathrm{d}3 \wedge \mathfrak{g}^* = 0$, the form 3 is specified up to a choice of 1, 2. In addition, d4, d5 determine 4, 5 (up to 1, 2), and have a common factor 3, hence 3 is now completely determined. As for d6, it just belongs to $\langle 1, 2\rangle \otimes \langle 4, 5\rangle$.

Furthermore, the only non-vanishing relations among exact 2-forms are $\mathrm{d}4 \wedge \mathrm{d}6 = \mp 1235$, $\mathrm{d}5 \wedge \mathrm{d}6 = 1234$. Let $x4 + y5$ be the candidate for fifth basis element. Then

$$\mathrm{d}(x4 + y5) \wedge \mathrm{d}6 = \mp x1235 + y1234 = 123(\mp x5 + y4)$$

from which $\mp x5 + y4$ is the new 4, so

$$\mathrm{d}(\mp x5 + y4) \wedge \mathrm{d}6 = \mp 123(x4 + y5).$$

In (4.18) instead, after renaming $e^6 = (z + a_1)e^6 + \lambda e^5$, we have

$$\mathrm{d}e^2 \wedge \mathrm{d}e^6 = \lambda e^{13}(-a_1 e^4)e^5, \qquad \mathrm{d}e^4 \wedge \mathrm{d}e^6 = \lambda e^1(-ze^2)e^{35};$$

so apparently $\langle 1, 2\rangle$ corresponds to $\langle e^1, e^3\rangle$, 3 should be e^5 and $\langle 4, 5\rangle$ is $\langle e^2, e^4\rangle$. Comparing

$$\mathrm{d}(-a_1e^4)\wedge \mathrm{d}e^6 = \lambda a_1 z e^{1235} \quad \text{and} \quad \mathrm{d}(-ze^2)\wedge \mathrm{d}e^6 = \lambda a_1 z e^{1345}$$

with the corresponding $\mathrm{d}4 \wedge d6 = \mp 1235$, $\mathrm{d}5 \wedge \mathrm{d}6 = 1234$, relative to **8**, **7**, shows that $\lambda a_1 z$ can be both positive or negative, so the algebra type is detected by the sign of this particular coefficient.

Corollary 4.9. *All nilpotent Lie algebras with Betti numbers* $b_1 = 2$, $b_2 = 4$

$$(0, 0, 12, 13, 23, 14), \qquad (0, 0, 12, 13, 23, 14+25), \qquad (0, 0, 12, 13, 23, 14-25)$$

bear an SU(3)-*structure fulfilling* (4.1).

It turns out quite instructive to provide a different proof of the result by exhibiting a change of basis. The detailed description given for **7**, **8** suggests that one should first exchange e^2 with e^3, then e^5 with e^3. Then redefine e^6 as the linear combination $(z+a_1)e^6 + \lambda e^3$ in order to have $(z+a_1)\mathrm{d}e^6 = a_1e^{14} - ze^{25}$. The algebra $(0, 0, -(z+a_1)e^{12}, -\lambda e^{13}, \lambda e^{23}, (z+a_1)a_1e^{14} - z(a_1+z)e^{25})$ is almost of the desired form. Forgiving the abuse of notation, a diagonal transformation

$$\begin{aligned} &e^1 \mapsto ae^1, \qquad e^2 \mapsto be^2, \qquad e^3 \mapsto ce^3, \qquad e^4 \mapsto fe^4,\\ &e^5 \mapsto ge^5, \qquad e^6 \mapsto he^6 \end{aligned} \tag{4.19}$$

allows enough freedom to detect the correct coefficients: for instance $\mathrm{d}e^3$ becomes $-(z+a_1)(c/ab)e^{12}$ etc., and normalisation entails

$$\begin{aligned} &-(z+a_1)c = ab, \quad -\lambda f = ac, \qquad \lambda g = bc\\ &(z+a_1)a_1h = af, \qquad -z(a_1+z)h = \pm bg. \end{aligned}$$

Thus c, f, g, h are determined by a, b and $z = \pm a_1b^2/a^2$. Taking $a = 1$ it is possible to assign a_1, z, find b accordingly, and so on with the remaining numbers.

Case 3. Since now each form in V_3 is closed, (4.9) determines $\mathrm{d}e^4 = -\lambda e^{15}$, and similarly (4.15) yields $z_1 = -a_1$ in (4.16). Recall that a_3 can be taken to be zero. Imposing $\mathrm{d}e^6$ to be closed further gives $z_3 = \lambda$, so one eventually arrives at

$$(0, \lambda e^{35}, 0, -\lambda e^{15}, 0, a_1(e^{14} - e^{23}) + \lambda e^{13}). \tag{4.20}$$

The cohomology depends on the value of a_1 as suspected. Indeed, the inspection of the differentials $\mathrm{d}e^{ij}$ says that $\dim \operatorname{Ker} \mathrm{d}|_{\Lambda^2} = 7$. Thus generically $b_2 = 5$ whilst $b_1 = 3$, giving a large range to choose from, in principle. But when a_1 vanishes, b_2 raises to 8 and the Lie algebra is

$$(0, 0, 0, 12, 13, 23)$$

in disguise. This corresponds to the example of a non-locally conformally flat SU(3)-structure carrying parallel torsion given in [34].

If a_1 is non-zero instead, two non-isomorphic real Lie algebras crop up

Corollary 4.10. *Of all nilpotent Lie algebras with* $b_1 = 3$, $b_2 = 5$ *only*

$$(0, 0, 0, 12, 23, 14 \pm 35)$$

possess half-integrable structures satisfying (4.1).

Proof. Assume $b_2 = 5$. First of all (4.20) is 3-step for all choices of a_1, which rules out some possibilities. Secondly the dimensions of the central descending series are 6, 3, 1, 0, so we are only left with **14**, **15**, **16** (the other series not distinguishing further).

Now we start playing around with bases: swapping e^2, e^5 allows to write

$$(0, 0, 0, -\lambda e^{12}, -\lambda e^{23}, \lambda e^{13} + a_1(e^{14} + e^{25})). \tag{4.21}$$

Consider the following relations relative to the Lie algebras on the right

$$\mathbf{14}\begin{cases} \mathrm{d}4 \wedge \mathrm{d}6 = 1235, \\ \mathrm{d}5 \wedge \mathrm{d}6 = 0, \end{cases} \qquad \mathbf{15}, \mathbf{16}\begin{cases} \mathrm{d}4 \wedge \mathrm{d}6 = \pm 1235, \\ \mathrm{d}5 \wedge \mathrm{d}6 = 1234. \end{cases}$$

These suggest to redefine e^4 by a linear combination $xe^4 + ye^5$. Then $\mathrm{d}(xe^4 + ye^5) \wedge \mathrm{d}e^6 = \lambda(-ya_1)e^{1234} + \lambda(-xa_1)e^{1235}$ implies $x = 1$, $y = 0$ hence that e^4 should remain unchanged, but e^5 itself should be rescaled by $-a_1e^5$. The product $d(-a_1e^5) \wedge \mathrm{d}e^6 = \lambda a_1^2 e^{1234}$ tells that (4.20) corresponds to

14	if and only if	$a_1 = 0$,
15, **16**	if	$a_1^2 = 1$.

While the vanishing of a_1 is excluded by assumption, $a_1 = \pm 1$ will transform (4.20) into either **16** or **15**. □

Explicitly, replacing e^4 in $\mathrm{d}e^6$ with $e^4 + ae^1 + be^3$ gives $\mathrm{d}e^6 = (\lambda + a_1 b)e^{13} + a_1(e^{14} + e^{35})$, simplified to $\mathrm{d}e^6 = a_1(e^{14} + e^{35})$ when $b = -\lambda/a_1$. Whilst **15** crops up once we set $1 = a_1$, we apply the transformation (4.19) to (4.21), and then normalise coefficients

$$-\lambda f = ab, \qquad -\lambda g = bc, \qquad a_1 h = af, \qquad -a_1 h = cg.$$

Choosing $a = 1$, $b = \lambda$, $c = 1$, tantamount as reversing the orientations of e^4, e^5 and rescaling e^2, e^6, eventually produces **16**.

We are now able to collect the outcome of all previous corollaries together

Theorem 4.11. *Up to isomorphism, there are precisely six real six-dimensional half-integrable nilpotent Lie algebras satisfying the* G_2T *equations*

$$(0, 0, 12, 13, 23, 14), \qquad (0, 0, 12, 13, 23, 14 - 25), \qquad (0, 0, 12, 13, 23, 14 + 25),$$
$$(0, 0, 0, 12, 23, 14 + 35), \qquad (0, 0, 0, 12, 23, 14 - 35), \qquad (0, 0, 0, 12, 13, 23)$$

with non-zero Lee form θ.

Observe that these involve only four different complex types. As the real parameter λ is fixed, each of (4.17), (4.18), (4.20) is defined in terms of effective coefficients, varying which changes the SU(3) reduction and possibly the Lie algebra. Therefore the complex types

$$(0,0,12,13,23,14) \qquad (0,0,0,12,23,14\pm 35) \qquad (0,0,0,12,13,23)$$

admit a one-parameter family of half-integrable SU(3)-structures, whilst

$$(0,0,12,13,23,14\pm 25) \tag{4.22}$$

possesses a real two-dimensional family of such.

Finally, note that the latter twin algebras, which have highest step-length and smallest Betti numbers, can degenerate to all others by an appropriate contraction limit [26]. As a typical example, we show the contracting procedure by means of which (0, 0, 12, 13, 23, 14) is attained as limit of (4.22). Introduce the real parameter t and choose the following basis for Λ^1 depending on t:

$$t^{-1}e^1, \quad te^2, \qquad e^3, \qquad t^{-1}e^4, \qquad te^5, \qquad t^{-2}e^6.$$

This leaves all differentials $\mathrm{d}e^i$ unchanged except for $\mathrm{d}e^6 = e^{14} \pm t^{-4}e^{25}$, which becomes e^{14} when $t \longrightarrow 0$. This is same as setting a_1 equal to zero in (4.18). The technique is also designed to increase Betti numbers, and produces the other Lie algebras of Theorem 4.11 in a natural way.

5. The derivative of the torsion

The behaviour of $\mathrm{d}T$ is examined here on a case-by-case basis. The Lie algebras (4.17) and (4.18) produce similar expressions

$$\mathrm{d}T = \tfrac{3}{2}\lambda^2\omega^2 - 2k^2e^{1256}, \qquad \mathrm{d}T = \tfrac{3}{2}\lambda^2\omega^2 - (a_1^2 + 2z^2)e^{1234},$$

both of which have a ω^2-component plus a non-vanishing extra term. For the algebras encountered in Case 3 no additional coefficient appears, hence the derivative of T has the simplest possible form

$$\mathrm{d}T = \tfrac{3}{2}\lambda^2\omega^2.$$

Lemma 5.1. *The Kähler form of the Lie algebras*

$$(0, \lambda e^{35}, 0, -\lambda e^{15}, 0, a_1(e^{14} - e^{23}) + \lambda e^{13})$$

is an eigenform of the Laplace–Beltrami operator Δ.

Proof. Since $d{*_6}d\omega = -(3/2)\lambda^2\omega^2$ we have $d^*d\omega = -{*_6}d{*_6}\,d\omega = 3\lambda^2\omega$. But the squared Kähler form ω^2 is closed and thus $d^*\omega = -{*_6}d((1/2)\omega^2) = 0$, so the Laplacian

$$\Delta = d^*d + dd^*$$

acts on ω just as the operator d^*d. □

Moreover, the following corollary is always true.

Corollary 5.2. *The exterior derivative of the torsion T is a form of type* 2, 2 *with respect to J.*

The dependence on the choice of one particular J vaguely reminds of the analogous situation in the quaternionic Kähler case, where a strong QKT structure is purposely defined by $dT \in \Lambda^{2,2}$, this time though with respect to the whole sphere of almost complex structures [33].

Recall that in terms of the fundamental G_2-representation V_7

$$T \in \Lambda^3 \cong \mathbb{R} \oplus V_7 \oplus \mathcal{S}^2_0 V_7, \qquad dT \in \Lambda^4 \cong \mathbb{R} \oplus V_7 \oplus \mathcal{S}^2_0 V_7.$$

Since T is a tensor attached to seven-dimensional geometry, a better counterpart is given by the following proposition.

Proposition 5.3.

$$dT \in \mathbb{R} \oplus \mathcal{S}^2_0 V_7 \subset \Lambda^4.$$

This improves the previous Corollary.

Proof. It is well-known that

$$V_7 = [\![\Lambda^{1,0}]\!] \oplus \mathbb{R}, \qquad \mathcal{S}^2_0 V_7 = [\![\mathcal{S}^{2,0}]\!] \oplus [\Lambda^{1,1}_0] \oplus [\![\Lambda^{1,0}]\!] \oplus \mathbb{R}$$

as irreducible SU(3) modules. Take a tangent vector X to M, so $X = \partial/\partial t + \hat{X}$ with $\hat{X} \in TN$. Then

$$\begin{aligned}(X \lrcorner *\varphi) \wedge \omega^2 &= (\tfrac{1}{2}\hat{X} \lrcorner\, \omega^2 - \psi^- + \hat{X} \lrcorner\, \psi^-\, dt)\omega^2 \\ &= (\tfrac{1}{2}\hat{X} \lrcorner\, \omega^2)\omega^2 - \psi^-\omega^2 + (\hat{X} \lrcorner\, \psi^-\, dt)\omega^2 = 0.\end{aligned}$$

The first two terms are 7-forms on a six-dimensional manifold, so zero, and the last vanishes by type. □

The proof of Lemma 5.1 leads to the following proposition.

Proposition 5.4. Assume (N, J, h) *is an almost Hermitian* 6*-manifold with a half-integrable* SU(3)*-structure* (ω, ψ^+). *If the* G_2*-structure* (2.2) *on* $N \times S^1$ *has closed torsion T, then the*

Kähler form is an eigenvector of the Laplace–Beltrami operator on N:

$$\Delta\omega = \tfrac{1}{2}\lambda^2\omega.$$

Proof. By (3.1) $\Delta\omega = \mathrm{d}^*\mathrm{d}\omega = -(\lambda^2/4)*_6\omega^2 = (\lambda^2/2)\omega$, so ω is an eigenform with eigenvalue $(1/2)\lambda^2$. □

This fact partly motivates the reason for attracting the attention to the following definition.

Definition 5.5 (Cleyton and Swann [13]). A *strong* G_2-manifold with torsion (SG_2T) is a G_2-manifold with a closed skew-symmetric torsion form.

Now, in the special case in which N is a complex manifold equation (2.10) seems very promising, for the derivative of the torsion becomes

$$\mathrm{d}T = -\mathrm{d}*_6\,\mathrm{d}\omega.$$

In other words if the G_2-structure φ is strong, then N satisfies

$$\mathrm{d}J\,\mathrm{d}\omega = 0,$$

recalling that the complex structure J and the Hodge operator agree on type $\{2,1\}$-forms. The significance of this equation was uncovered in [3], then exploited in [21], together with the following fact of the utmost importance:

Proposition 5.6 (Alexandrov and Ivanov [3]). *A Hermitian manifold* (N, J, h) *of dimension greater than four is* SKT *only if it is not balanced, i.e. if the Lee form* ϑ *differs from zero*.

As $J\vartheta = \mathrm{d}^*\omega$, the SKT notion is antithetical to that of cosymplectic SU(3)-structures, and this conflict has dramatic consequences, in the light of the fact [21] that the holonomy of an invariant Hermitian structure on a nilmanifold N of dimension n reduces to SU(n) exactly when the metric is balanced. It basically erases all chances for a strong G_2-structure with torsion on $N \times S^1$ to induce an SKT structure on N.

Recall that when J is integrable, exterior differentiation is just $\partial + \bar{\partial}$, so $\mathrm{d}*_6\,\mathrm{d} = 2\mathrm{i}\bar{\partial}\partial$ and the SKT condition is the same as the $\partial\bar{\partial}$-lemma. But the six-dimensional Lee form is essentially β, and asking M^7 to be strong and J a complex structure, forces N to be Kähler (cf. Proposition 3.2). Compare this to the result that a compact, conformally balanced manifold with SU(n) holonomy satisfying $\mathrm{d}J\,\mathrm{d}\omega = 0$ is in fact Calabi-Yau, see [41].

This is confirmed by the fact that it is not possible to obtain strong G_2 metrics from the Lie algebras we have considered, as no nilpotent Lie algebra from Theorem 4.11 generates a structure with strong torsion on products, unless $\lambda = 0$. Because

$$\mathrm{d}(JV_3) \subseteq \Lambda^2 V_3 \oplus (V_3 \wedge JV_3)$$

in all instances, the common offending term in dT has the expected coefficient $-\lambda^2/4$ only in the degenerate case.

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